AD-A120 569 SEQUENTIAL PARAMETER ESTIMATION IN EXPONENTIAL AUTOREGRESSIVE PROCESSES. (U) AEROSPACE CORP EL SEGUNDO CA GUIDANCE AND CONTROL DIV M R CHERNICK ET AL. 1/1

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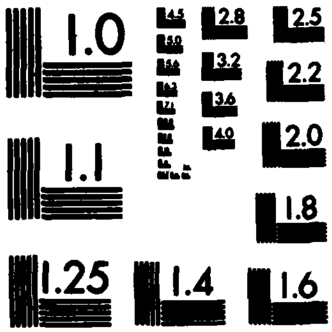

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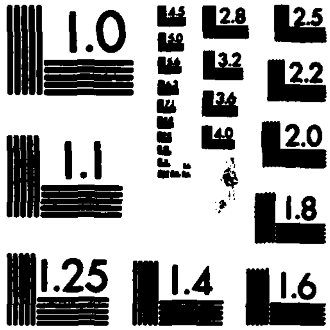

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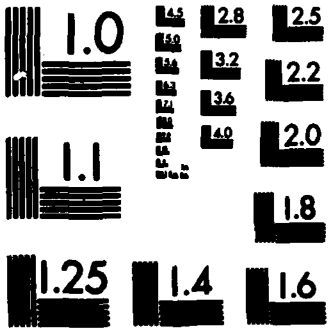

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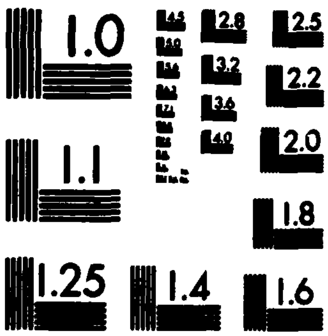

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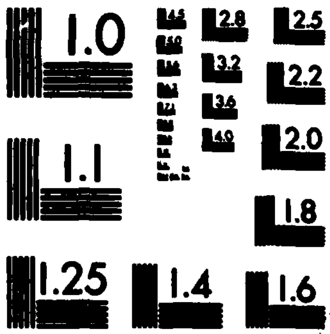

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This report was submitted by The Aerospace Corporation, El Segundo, CA 90245, under Contract No. F04701-81-C-0082 with the Space Division, P. O. Box 92960, Worldway Postal Center, Los Angeles, CA 90009. It was reviewed and approved for The Aerospace Corporation by C. M. Price, Director, Satellite Navigation Department. Captain James C. Garcia, SD/YLXS, was the Deputy for Technology project engineer.

This technical report has been reviewed and is approved for publication. Publication of this report does not constitute Air Force approval of the report's findings or conclusions. It is published only for the exchange and stimulation of ideas.

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REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: SD-TR-82-66	2. GOVT ACCESSION NO.: AD-A120569
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle): SEQUENTIAL PARAMETER ESTIMATION IN EXPONENTIAL AUTOREGRESSIVE PROCESSES	5. TYPE OF REPORT & PERIOD COVERED
	6. PERFORMING ORG. REPORT NUMBER: TR-0082(9990)-2
7. AUTHOR(s): M. R. Chernick and V. K. Murthy	8. CONTRACT OR GRANT NUMBER(s): F04701-81-C-0082
9. PERFORMING ORGANIZATION NAME AND ADDRESS: The Aerospace Corporation, El Segundo, Calif. 90245	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS: Space Division, Air Force Systems Command, Los Angeles, Calif. 90009	12. REPORT DATE: 15 Sep 82
	13. NUMBER OF PAGES: 13
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report): Unclassified
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

autoregressive processes
sequential estimation
exponential distribution

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This paper generalizes the sequential estimation of autoregressive parameters as given in Gaver and Lewis (1980) to pth order processes EAR(p) and GAR(p). The first two moments of the stopping time distribution are computed for the EAR(2) process, a procedure for estimating the parameter λ is given, and a simulation is given to show how large the stopping time can be for various α_1 and α_2.

lambda

alpha

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1. INTRODUCTION

A stationary first-order autoregressive process with exponential marginal distributions was defined in Gaver and Lewis (Ref. 1). The process is defined by the following recursion:

$$X_n = \rho X_{n-1} + I_n E_n \qquad \text{where } 0 < \rho < 1 \tag{1}$$

and $I_n = 0$ with probability ρ, $I_n = 1$ with probability $1 - \rho$, and $\{E_n\}$ is an independent identically distributed (i.i.d.) sequence of exponential random variables with rate parameter λ. The $\{I_n\}$ sequence is also i.i.d. and is independent of $\{E_n\}$.

The sequence is a special case of the EARMA(1,1) sequence studied in Jacobs and Lewis (Ref. 2). Properties of the sequence are given in Gaver and Lewis (Ref. 1), Jacobs and Lewis (Ref. 2), and Chernick (Ref. 3). A relationship between the sequence (which is denoted by EAR(1)) and another Markovian exponential process (Tavares (Ref. 4)) is pointed out in Chernick, et. al. (Ref. 5).

In section 5 of Gaver and Lewis (Ref. 1) it is observed that when $\rho > 0$ it is possible to determine ρ exactly. If we let $Z_n = X_{n+1}/X_n$, we see that $Z_n = \rho$ when I_n is zero. Because $P(I_n = 0) = \rho > 0$, I_n will be zero infinitely often. By waiting for the first repeated value of Z_n, we find that ρ is equal to this repeated value. The quantity ρ will also be the minimum value for Z_n because

$$Z_n = \rho + \frac{I_{n+1} E_{n+1}}{X_n} \tag{2}$$

and

$$\frac{I_{n+1} E_{n+1}}{X_n} > 0 \text{ for each } n.$$

We note that Z_n has a continuous distribution when $I_{n+1} = 1$ and has probability ρ concentrated at ρ. So the only value that will repeat in the sequence is ρ. In practice other values will have a small probability of occurrence due to the discreteness of the random number generator.

Gaver and Lewis (Ref. 1) point out that if we use the stopping time

$$T = \min \{ n : Z_n \text{ repeats its previous minimum value} \}$$

then T is the sum of two geometric random variables plus one. So T has expectation $1 + (2/\rho)$ and variance $2(1-\rho)/\rho^2$. Clearly if ρ is not too small the expected value of T and its variance will be small. In fact, it is easy to determine the exact probability distribution for T,

$$\begin{aligned} P(T=n) &= (n-2)\ \rho^2\ (1-\rho)^{n-3} && \text{for } n \geq 3 \\ &= 0 && \text{for } n < 3. \end{aligned} \tag{3}$$

From Eq. (3) it is easy to determine that

$$\begin{aligned} P(T>n) &= (1+(n-3)\rho\ )(1-\rho)^{n-3} && \text{for } n \geq 4 \\ &= 1 && \text{for } n \leq 3. \end{aligned} \tag{4}$$

So as long as ρ is not very small it is unlikely that T will be very large. On the other hand, if there is a possibility that ρ is small and one cannot afford to take more than, say, n_0 samples, we would recommend using the stopping time T' where

$$T' = \min(T,\ n_0).$$

When $T > n_0$ the logical choice for an estimate of ρ is

$$\hat{\rho} = \min \{ Z_n : n \leq T' \}.$$

The estimator $\hat{\rho}$ is greater than or equal to ρ and the bias will be small for reasonably large n.

Lawrance and Lewis (Ref. 6) have generalized the EARMA model to higher order autoregressive and moving average terms. In particular, they define the EAR(p) processes as follows

$$X_i = \begin{Bmatrix} \alpha_1 X_{i-1} & \text{with probability } a_1 \\ \alpha_2 X_{i-2} & \text{with probability } a_2 \\ \vdots & \vdots \\ \alpha_p X_{i-p} & \text{with probability } a_p \end{Bmatrix} + \varepsilon_i \tag{5}$$

where $$a_1 = (1-\alpha_2) \qquad a_p = \prod_{j=2}^{p} \alpha_j$$

and $a_\ell = \left(\prod_{j=2}^{\ell} \alpha_j\right)(1-\alpha_{\ell+1})$, $\ell = 2, \ldots, p-1$, $1 > \alpha_i > 0$

for $i=1,2, \ldots, p$

and ε_i has the distribution required so that X_i has an exponential distribution with parameter λ for each i. For $p > 2$ the requirement that such an i.i.d. sequence exists imposes additional constraints on the parameters. Lawrance and Lewis derive the distribution for ε_i explicitly only in the case $p=2$.

Gaver and Lewis (Ref. 1) showed that for the EAR(1) process. once ρ has been determined through the sequential estimation procedure, the E_i's can be recovered exactly for $i>2$. Because the sequence $\{E_n\}$ is i.i.d. exponential with the rate parameter λ, the usual maximum likelihood estimates for λ can be determined. In section 2 of this report, it is demonstrated that a generalization of the sequential stopping rule can be used to explicitly determine the α_i's for each i. Section 3 discloses how a conditional likelihood estimator can be determined for λ. For $p>2$ the non-zero ε_i's cannot all be recovered and hence the generalization of the result for $p=1$ is not straightforward. Explicit results are obtained for the case $p=2$.

2. DETERMINING THE AUTOREGRESSIVE PARAMETERS

Because the required distribution for the $\{\varepsilon_n\}$ sequence always has positive probability concentrated at zero, it is possible to determine $\alpha_1, \alpha_2, \ldots, \alpha_p$ by keeping track of the ratios

$\frac{X_i}{X_{i-1}}, \frac{X_i}{X_{i-2}}, \ldots, \frac{X_i}{X_{i-p}}$. Once the value of $\frac{X_i}{X_{i-k}}$ is repeated, that repeated value is α_k. The stopping time T is then the smallest n such that $\frac{X_i}{X_{i-k}}$ has been repeated for all $k = 1,2, \ldots, p$.

For the case when $p = 2$, we shall determine the distribution of T, its expectation and variance.

The EAR(2) process of Lawrance and Lewis is given as follows:

$$X_i = \begin{Bmatrix} \alpha_1 X_{i-1} & \text{with probability } 1 - \alpha_2 \\ \alpha_2 X_{i-2} & \text{with probability } \alpha_2 \end{Bmatrix} + \varepsilon_i$$

where

$$\varepsilon_i = \begin{Bmatrix} 0 & \text{with probability } \alpha_1/(1 + \alpha_1 - \alpha_2) \\ E_i & \text{with probability } (1 - \alpha_1)(1 - \alpha_2)/(1 - \delta) \\ \delta E_i & \text{with probability } (1 - \alpha_2)(\alpha_1 - \alpha_2)^2/\{(1 + \alpha_1 \\ & \quad - \alpha_2)(1 - \delta)\} \end{Bmatrix}$$

and $\delta = (1 + \alpha_1 - \alpha_2)\,\alpha_2$ and $\{E_i\}$ is an i.i.d. exponential sequence with parameter λ.

Let $T_1 = \min\left\{n\colon \frac{X_i}{X_{i-1}} \text{ is the same for two values of } i < n\right\}$ and $T_2 = \min \left\{n\colon \frac{X_i}{X_{i-2}} \text{ is the same for two values of } i \leq r\right\}$. Then let $T = \max\{T_1, T_2\}$.

Now $X_i = \alpha_1 X_{i-1}$ with probability $P_1 = \alpha_1(1 - \alpha_2)/(1 + \alpha_1 - \alpha_2)$ and $X_i = \alpha_2 X_{i-2}$ with probability $P_2 = \alpha_1\alpha_2/(1 + \alpha_1 - \alpha_2)$.

We consider the stochastic sequence $\{y_i\}_{i=1}^n$ where $y_i = 0$, 1 or 2. The y_i's are independent random variables with $P[y_i = 1] = P_1$, $P[y_i = 2] = P_2$ and $P[y_i = 0] = 1 - P_1 - P_2$. Let V be the first time that both 1 and 2 are repeated. Clearly $T = V + 1$.

Simple combinatorial arguments show that $P[V = n + 1] = P[T = n + 2] =$ $n\{P_1^2[(1 - P_1)^{n-1} - (1 - P_1 - P_2)^{n-1}] + P_2^2[(1 - P_2)^{n-1} - (1 - P_1 - P_2)^{n-1}]\} -$ $n(n - 1) P_1P_2 (P_1 + P_2)(1 - P_1 - P_2)^{n-2}$ for $n \geq 4$. $E(T) = 1 + E(V)$ and $Var(T) = Var(V)$. Computations show

$$E(T) = 1 + \frac{2}{P_1} + \frac{2}{P_2} - \frac{2}{(P_1 + P_2)} - \frac{2P_1P_2}{(P_1 + P_2)^3}$$

and

$$Var(T) = \frac{2}{P_1^2} - \frac{2}{P_1} + \frac{2}{P_2^2} - \frac{2}{P_2} + \frac{2}{(P_1 + P_2)} - \frac{2}{(P_1 + P_2)^2} + \frac{2\, P_1P_2}{(P_1 + P_2)^3}$$
$$- \frac{32\, P_1\, P_2}{(P_1 + P_2)^4} - \frac{4\, P_1\, P_2^2}{(P_1 + P_2)^6}.$$

For $\alpha_1 = 0.5$ and $\alpha_2 = 0.4$, $E(T) = 13.86$, whereas for the EAR(1) process with $\rho = 0.5$ $E(T) = 5$, so $E(T)$ increases significantly as the order of the process increases. In principle the distribution of T can be determined for any order p but apparently the distribution becomes more complicated. Clearly $E(T)$ grows as the order is increased and probably $Var(T)$ also grows as the order is increased. For higher order models it may be necessary to truncate the stopping time. However, it is not clear how one would estimate the α_i's which have not been determined by repetition.

The gamma first order autoregressive process of Gaver and Lewis (Ref. 1) (GAR(1)) can be generalized to higher order models in the same way that Lawrance and Lewis generalized the EAR(1) process. In fact, the GAR(p) process can be thought of as the sum of k EAR(p) processes when the parameter k is an integer. It is obtained in the following way:

Let $X_{n1}, X_{n2}, \ldots, X_{nk}$ be k EAR(p) processes each with the same parameters $\alpha_1, \alpha_2, \ldots, \alpha_p$, and λ. The processes are related in that if $X_{n1} = \alpha_r X_{(n-1)1} + \varepsilon_{n1}$ then $X_{n2} = \alpha_r X_{(n-1)2} + \varepsilon_{n2}, \ldots,$ $X_{np} = \alpha_r X_{(n-1)p} + \varepsilon_{np}$ for $r = 1, 2, \ldots, p$. The sequences $\{\varepsilon_{n1}\}, \{\varepsilon_{n2}\}, \ldots, \{\varepsilon_{np}\}$ are independent. Define

$$S_n = \sum_{i=1}^{k} X_{ni}.$$

Then S_n is a GAR(p) process and just as in Eq. (5) we have

$$S_i = \begin{bmatrix} \alpha_1 \ S_{i-1} & \text{with probability } a_1 \\ \alpha_2 \ S_{i-2} & \text{with probability } a_2 \\ \vdots & \vdots \\ \alpha_p \ S_{i-p} & \text{with probability } a_p \end{bmatrix} + e_i$$

where

$$a_1 = (1 - \alpha_2),\ a_p = \prod_{j=2}^{p} \alpha_j,\ a_\ell = \prod_{j=2}^{\ell} \alpha_j \ (1 - \alpha_{\ell+1}) \tag{6}$$

$$\ell = 2, \ldots, p-1,\ 1 > \alpha_i > 0 \ \text{ for } i = 1, 2, \ldots, p$$

and

$$e_i = \sum_{j=1}^{k} \varepsilon_{ij}.$$

The determination of the α_i's for the GAR(p) process is the same as for the EAR(p) process. Because the stopping time T depends only on the sequence $\{y_i\}_{i=1}^{n}$ when $p = 2$, the stopping time for the GAR(2) process has the same distribution as for the EAR(2) process.

3. ESTIMATION OF LAMBDA

Once the α_i's have been determined for $i=1,2, \ldots, p$ we can compute the following residuals: $r_{i1} = X_i - \alpha_1 X_{i-1}$, $r_{i2} = X_i - \alpha_2 X_{i-2} \ldots r_{ip} = X_i - \alpha_p X_{i-p}$. When $r_{ij} = 0$ for some j this indicates that $\epsilon_j = 0$ and $X_i = \alpha_j X_{i-j}$. We can then determine a conditional likelihood for the residuals given the set of ϵ_i which are zero. For those i for which $r_{ij} \neq 0$ for any j, ϵ_i is greater than zero.

We shall now consider the case p = 2 for simplicity. Let $I_1 = [i: \ r_{i1} > 0$ and $r_{i2} > 0]$, $I_2 = [i: \ r_{i1} < 0$ and $r_{i2} > 0]$ and $I_3 = [i: \ r_{i1} > 0$ and $r_{i2} < 0]$ and let

$$I = \bigcup_{j=1}^{3} I_j.$$

Let k be the number of elements in I. When $i \epsilon I_2$, $\epsilon_i = r_{i2} > 0$, and when $i \epsilon I_3$, $\epsilon_i = r_{i1} > 0$. If $i \epsilon I_1$ we do not know whether $\epsilon_i = r_{i1}$ or $\epsilon_i = r_{i2}$. For $i \epsilon I_2$ the conditional likelihood is $\lambda e^{-\lambda r_{i2}}$ and for $i \epsilon I_3$ it is $\lambda e^{-\lambda r_{i1}}$. When $i \epsilon I_1$ we only know that $\epsilon_i > 0$ and is either r_{i1} or r_{i2}.

Consequently, the conditional likelihood is

$$\lambda \left[(1 - \alpha_2) e^{-\lambda r_{i1}} + \alpha_2 e^{-\lambda r_{i2}} \right].$$

Because the ϵ_i's are independent given that they are not zero, the conditional likelihood L_c is given by

$$L_c = \lambda^k \prod_{i \epsilon I_1} \left[(1 - \alpha_2) e^{-\lambda r_{i1}} + \alpha_2 e^{-\lambda r_{i2}} \right] \prod_{i \epsilon I_2} e^{-\lambda r_{i2}} \prod_{i \epsilon I_3} e^{-\lambda r_{i1}}.$$

Maximizing L_c yields a conditional maximum likelihood estimator and generalizes the method of Gaver and Lewis. In general, for the pth order process there will be 2^p-1 sets I_ℓ to consider since it is possible for r_{it} to be less than zero or greater than zero for each i and t but r_{it} cannot be less than zero for i fixed and each t.

Let

$$I = \bigcup_{\ell=1}^{2^p-1} I_\ell$$

and k be the number of elements in I. Then the conditional likelihood is

$$L_c = \lambda^k \prod_{i\in I_1} \left[\sum_{j=1}^{p} p_{ij}\ e^{-\lambda r_{ij}}\right] \prod_{i\in I_2} \sum_{j=1}^{p}\left[p_{ij}\ e^{-\lambda r_{ij}}\right]\cdots$$
$$\prod_{i\in I_{2^p-1}} e^{-\lambda r_{ip}}$$

The p_{ij}'s depend on which r_{ij}'s are less than zero and in particular $p_{ij} = 0$ if $r_{ij} < 0$.

Another reasonable way to estimate λ when $p = 2$ is obtained by considering the following equation:

$$X_\ell - (\alpha_1\gamma_\ell X_{\ell-1} + \alpha_2(1-\gamma_\ell)\ X_{\ell-2}) = \varepsilon_\ell \text{ for } \ell = 3,4\ ,\ldots,\ n$$

where

$$P(\gamma_\ell = 0) = \alpha_2 = 1 - P\ (\gamma_\ell = 1) \quad \text{and the sequence } \{\gamma_\ell\} \tag{7}$$

is i.i.d. and independent of $X_1,\ X_2,\ \ldots,\ X_{n-1}$.

Because $E(\varepsilon_\ell) = 1/\lambda$ and $E(\gamma_\ell) = 1 - \alpha_2$,

$$\frac{1}{n-3}\sum_{j=3}^{n}\{X_j - \alpha_1\ (1-\alpha_2)\ X_{j-1} - \alpha_2^2\ X_{j-2}\}$$

is an unbiased and consistent estimate for $1/\lambda$. Consequently

$$(n-3)/\sum_{j=3}^{n}\left[X_j - \alpha_1 (1-\alpha_2) X_{j-1} - \alpha_2^2 X_{j-2}\right]$$

will be a consistent estimator for λ.

For the GAR(p) with k known, the conditional likelihood approach could be employed. The simple approach given in the preceding paragraph can also be used. $E(e_n) = k/\lambda$ and so the estimator

$$k(n-3)/\sum_{j=3}^{n}\{S_j - \alpha_1 (1-\alpha_2) S_{j-1} - \alpha_2^2 S_{j-2}\}$$

will be a consistent estimator for λ.

4. CONCLUSIONS

For low order EAR or GAR models this sequential parameter estimation procedure provides a satisfactory way of determining the α_i's and then estimating λ. Table 1 shows results of simulating the process for various values of α_1 and α_2. The stopping time T is replicated fifty times and the sample mean $(\bar{T})$, sample variance (S^2) are compared with their theoretical values E(T) and V(T) respectively. Also, the maximum value of $T(T_{max})$ is given for each α_1 and α_2. Even for small values of the α_i's it may be possible to use the stopping time because T is unlikely to exceed 500. Considering the X_i's to be the interarrival times for a point process, the EAR process introduces a correlation structure to the time between events and hence generalizes the Poisson process. Generalizations of the Poisson process are important because they allow for greater flexibility in modelling series of failure times. These models can then be used to assess the risks associated with rare catastrophic events such as an accident at a nuclear power plant.

Table 1. Simulation of the EAR(2) Process

α_1	α_2	$E(T)$	$\bar{T}$	$V(T)$	S^2	T_{max}
0.2	0.1	111.12	113.64	5865.9	5932.9	356
0.2	0.2	51.90	46.34	1096.7	702.8	123
0.2	0.3	32.95	34.06	320.5	407.3	85
0.2	0.4	24.41	22.58	106.9	195.0	46
0.4	0.1	66.07	68.18	2021.6	2033.5	232
0.4	0.2	31.53	29.56	382.6	363.0	85
0.4	0.3	20.53	20.30	112.1	163.7	64
0.4	0.4	15.63	14.60	36.3	36.7	37
0.5	0.1	57.14	61.48	1492.7	1999.5	232
0.5	0.2	27.48	27.96	283.8	342.0	85
0.5	0.3	18.05	18.80	83.2	140.4	51
0.5	0.4	13.88	13.00	26.5	27.1	29

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